

2D(1+1) Quantum Gravity. Gravitational Quantum Stationary Hamilton-Jacobi Equation

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Abstract

In the present article, we construct a 2D formulation of quantum gravity in the framework of a deterministic theory. In this context, a Quantum stationary Hamilton-Jacobi equation is derived from the Klein-Gordon equation written in the presence of a gravitational field. We show that this equation reduces to the Quantum stationary Hamilton-Jacobi equation when the gravitational field is not present in the 2D time-space. As a second step, we introduce the quantum gravitational Lagrangian for the quantum motion of a particle moving in the presence of a gravitational field. We, deduce the relationship between the gravitational quantum conjugate momentum and the velocity of the particle.

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1. Introduction

Since three years, a deterministic approach of quantum mechanics is presented in Refs. [1, 2, 3, 4, 5, 6, 7, 8] as an alternative of the standard quantum mechanics (Copenhagen interpretation). It consists on the introduction of a Quantum Stationary Hamilton-Jacobi Equation (QSHJE) already established by Faraggi and Matone [9, 10], Floyd [11], Bohm [12] and de Broglie [13]

$$\frac{1}{2m_0} \left(\vec{\nabla} S_0(\vec{r}) \right)^2 - \frac{\hbar^2}{2m_0} \frac{\Delta R(\vec{r})}{R(\vec{r})} + (E - V(\vec{r})) = 0 \quad , \quad (a)$$

$$\vec{\nabla} \cdot (R^2(\vec{r}) \vec{\nabla} S_0) = 0 \quad , \quad (b) \quad (1)$$

which correspond in classical mechanics ($\hbar \rightarrow 0$) to the Classical Stationary Hamilton-Jacobi Equation (CSHJE)

$$\frac{1}{2m_0} \left(\vec{\nabla} S_0(\vec{r}) \right)^2 + (E - V(\vec{r})) = 0 \quad . \quad (2)$$

R and S_0 being real functions and S_0 representing the action of the quantum system.

Before studying the 3D systems, we have investigated the 1D problems, so, we took up the 1d QSHJE written as [1, 11, 14]

$$\begin{aligned} \frac{1}{2m_0} \left(\frac{dS_0}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \left[\frac{3}{2} \left(\frac{dS_0}{dx} \right)^{-2} \left(\frac{d^2 S_0}{dx^2} \right)^2 - \right. \\ \left. \left(\frac{dS_0}{dx} \right)^{-1} \left(\frac{d^3 S_0}{dx^3} \right) \right] + V(x) = E \quad . \end{aligned} \quad (3)$$

Taking advantage of the solution of the QSHJE (Eq. (3)) [1, 11, 14]

$$S_0(x) = \hbar \arctan \left(a \frac{\theta(x)}{\phi(x)} + b \right) \quad , \quad (4)$$

we introduced a quantum Lagrangian of the form [1]

$$L(x, \dot{x}, a, b) = \frac{1}{2} m \dot{x}^2 f(x, a, b) - V(x) \quad , \quad (5)$$

and derived the dynamical relation giving the velocity of the particle in function of the energy, the potential and the quantum conjugate momentum [1]

$$\dot{x} \frac{\partial S_0}{\partial x} = 2(E - V) , \quad (6)$$

Using this relation, we derived the first integral of the Quantum Newton's Law [1], and plotted quantum trajectories of particles moving under different potentials [2]. Later, we presented a generalization of our approach to the relativistic systems [4, 5] and spinning particles [6]. For the quantum relativistic systems we introduced the Relativistic QSHJE (RQSHJE) [4]

$$\begin{aligned} \frac{1}{2m_0} \left(\frac{\partial S_0}{\partial x} \right)^2 - \frac{\hbar^2}{4m_0} \left[\frac{3}{2} \left(\frac{\partial S_0}{\partial x} \right)^{-2} \left(\frac{\partial^2 S_0}{\partial x^2} \right)^2 - \right. \\ \left. \left(\frac{\partial S_0}{\partial x} \right)^{-1} \left(\frac{\partial^3 S_0}{\partial x^3} \right) \right] + \frac{1}{2m_0 c^2} [m_0^2 c^4 - (E - V)^2] = 0 , \end{aligned} \quad (7)$$

and construct the Lagrangian of height energy particles as follows [4]

$$L(x, \dot{x}, a, b) = -m_0 c^2 \sqrt{1 - f(x, a, b) \frac{\dot{x}^2}{c^2}} - V(x) . \quad (8)$$

The solution of Eq. (7) can be expressed by Eq. (4), where θ and ϕ represent now two real independent solutions of the Klein-Gordon equation

$$-c^2 \hbar^2 \frac{\partial^2 \phi}{\partial x^2} + [m_0^2 c^4 - (E - V)^2] \phi(x) = 0 . \quad (9)$$

Applying the least action principle to the Lagrangian (8) and after some calculations we leads to the dynamical relation [4]

$$\dot{x} \frac{\partial S_0}{\partial x} = E - V(x) - \frac{m_0^2 c^4}{(E - V)} , \quad (10)$$

from which we derived the relativistic quantum Newton's Law [4]. Then, in Ref. [5], we plotted the relativistic quantum trajectories for different potentials.

The spinning particles are also studied in the context of our deterministic approach of quantum mechanics. So, beginning with the 1D Dirac's equation [6]

$$-i\hbar c \sigma_x \frac{d\psi}{dx} = (E - V(x) - \sigma_z m_0 c^2) \psi \quad (11)$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (12)$$

we established two RQSHJES $_{\frac{1}{2}}$, which can be written as [6]

$$\begin{aligned} & \frac{1}{2m_0} \left(\frac{dS_{m_s}}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \{S_{m_s}, x\} + \frac{\hbar^2}{2m_0} (E - V + (-1)^{2m_s+1} m_0 c^2)^{\frac{1}{2}} \\ & \cdot \frac{d^2}{dx^2} \left[(E - V + (-1)^{2m_s+1} m_0 c^2)^{-\frac{1}{2}} \right] + \frac{1}{2m_0 c^2} \left[m_0^2 c^4 - (E - V)^2 \right] = 0, \end{aligned} \quad (13)$$

where S_{m_s} represents the quantum reduced action of the particle with spin $\frac{1}{2}$ corresponding to the projection $m_s = \pm \frac{1}{2}$ of the spin [6]. The quantity

$$\{S_{m_s}, x\} = \left[\frac{3}{2} \left(\frac{dS_{m_s}}{dx} \right)^{-2} \left(\frac{d^2 S_{m_s}}{dx^2} \right)^2 - \left(\frac{dS_{m_s}}{dx} \right)^{-1} \left(\frac{d^3 S_{m_s}}{dx^3} \right) \right]$$

represents the Schwarzian derivative of S_{m_s} with respect to x .

All the above results are presented for the 1D spaces. However, the physical phenomena happens in the 3D spaces. This is the reason why we present in Refs. [7, 8] a generalization to 3D of our deterministic approach. First, we have presented a solution of the 3D QSHJE (Eqs. (1)) and given the reduced action by the relation (4), but where θ and ϕ are two real independent solutions of the 3D Schrödinger equation. Secondly, we introduced a 3D quantum Lagrangian [7]

$$L_q = \frac{m_0}{2} \dot{x}^2 a_{xx}(\vec{r}) + \frac{m_0}{2} \dot{y}^2 a_{yy}(\vec{r}) + \frac{m_0}{2} \dot{z}^2 a_{zz}(\vec{r}) - V((\vec{r})), \quad (14)$$

from which we derived the dynamical relation connecting between the quantum conjugate momentum and the velocity of the particle [6]

$$\vec{v} \cdot \nabla \vec{S}_0 = 2 [E - V(\vec{r})]. \quad (15)$$

In Eq. (14), the quantities $a_{\mu\mu}$ represents the diagonal components of the quantum metric tensor which define the curvature of the space by the quantum potential. The components $a_{\mu\mu}$ play the same role as the function $f(x)$ present in Eq. (5) for the 1D systems. Using Eq. (15) after taking advantage on the expression of the reduced action $S_0(\vec{r})$ (Eq. (4)) and the solutions of the 3D Schrödinger equation, we plot the quantum trajectories of the Hydrogen's electron for different states [7].

Remark that in our formulation, we obtain easily the classical equations only by taking $\hbar \rightarrow 0$ in the quantum equations (1-15). This is an important result, since we build a deterministic quantum mechanics which, at the classical limit ($\hbar \rightarrow 0$), leads to the deterministic classical mechanics [1, 2, 3, 4, 5, 7, 8].

After generalizing our approach to the relativistic systems, the spinning particles and the 3D spaces, an interesting questions arise. It concerns the gravity interaction. Is it possible to study the motion of a quantum particle in the presence of a gravitational field? If yes, how it should be made? The answer of these questions may construct a deterministic approach of quantum gravity.

Since the appearance of quantum mechanics many theories of quantum gravity are developed in the context of the standard quantum mechanics ie the probabilistic approach of quantum phenomena. String theory, Dilaton theories and String-dilaton theories are the most important examples of such theories [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. In spite of their theoretical results, many difficulties stand in the way of these theories. One of these difficulties consist on the fact that these theories connect between a probabilistic quantum theory and a deterministic theory of gravity (General Relativity). This difficulty is a conceptual one and not mathematical. In fact, the gravitational field which is a manifestation of a mass M present in a determined space (Newton's theory of gravity, General Relativity) is now described by a quantum theory which stipulate that the mass M do not occupy a well determined space but is spread over all the space

according to a probabilistic law. However, when we take the classical limit, one should find the deterministic theory of gravitation (GR), so, we don't know how a probabilistic theory should reduce to a deterministic one at the classical limit. Thus, in our opinion, it is necessary to build a deterministic quantum gravity theory connecting between two deterministic theories, GR and a deterministic quantum mechanics. For this aim, we propose in this article to connect our deterministic approach of quantum mechanics and the general relativity.

Now, let us remind the most results of general relativity [30, 31]. After announcing the equivalence postulate, Einstein deduce that in order to generalize the Poisson equation of the gravitational field Φ

$$\Delta\Phi - 4\pi G\mu = 0 \quad (16)$$

to the relativistic theory, the space must be curved. After setting the interval ds as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \mu, \nu = 0, 1, 2, 3, \quad (17)$$

Einstein obtained the following field equations [30, 31]

$$E_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3 \quad (18)$$

G being the universal gravitational constant, μ is the mass density and $g_{\mu\nu}$ is the metric tensor of the curved space. $T_{\mu\nu}$ is the momentum-energy tensor. $R_{\mu\nu}$ is the Ricci tensor given as [30, 31]

$$R_{\mu\nu} = \partial_\gamma \Gamma_{\mu\nu}^\gamma - \partial_\nu \Gamma_{\mu\gamma}^\gamma + \Gamma_{\mu\nu}^\gamma \Gamma_{\gamma\delta}^\delta - \Gamma_{\mu\gamma}^\delta \Gamma_{\nu\delta}^\gamma. \quad (19)$$

$\Gamma_{\mu\nu}^\gamma$ are the Christoffel symbols

$$\Gamma_{\mu\nu}^\gamma = \frac{1}{2}g^{\gamma\delta} \left(\frac{\partial g_{\delta\mu}}{\partial x^\nu} + \frac{\partial g_{\delta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\delta} \right). \quad (20)$$

R is the curvature invariant

$$R = g^{\mu\nu} R_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3. \quad (21)$$

The free particle describe a trajectory given by the geodesic equations [30, 31]

$$\frac{d^2 x^\sigma}{ds^2} + \Gamma_{\mu\nu}^\sigma \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \quad \mu, \nu = 0, 1, 2, 3. \quad (22)$$

In fact, the last equation represents the law of motion under the gravitational field and can be derived from the Einstein-Hilbert action [30, 31]

$$S(x^\mu) = -m_0 c \int ds = -m_0 c \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} ds \quad (23)$$

after using the least action principle. ds is the interval given in Eq. (17) and is considered as an evolution parameter, and $\dot{x}^\mu = dx^\mu/ds$.

The results of the GR will be taking into consideration when we try to construct the quantum gravity formulation. However, a construction of a quantum gravity theory in 4D seems to be so complicated. Then, we propose to construct a 2D formulation of quantum gravity. In this order, we should first investigate the 2D classical gravity, and determine the metric of the 2D space. We will see in Sec. 2 the difficulties we face to construct such a gravity theory. In Sec. 3, in spite of the difficulties we face in Sec. 2, we derive a Gravitational QSHJE (GQSHJE) from the 2D Klein-Gordon equation written in the presence of the gravitational field. After, in Sec. 4, we introduce a Lagrangian from which we derive the equation of motion of a free particle for a stationary system and a static metric. Finally, in Sec. 5, we introduce the Einstein-Hilbert action and derive the quantum gravitational law of motion for a non-stationary system and a general metric.

2. General relativity and 2D Gravity

In 2-dimensions, the Einstein's gravitational theory is trivial. This is due to the fact that the Einstein tensor $E_{\alpha\beta}$ (Eq. (18)) is identically zero for all 2-dimensional metrics. As a consequence, the Einstein equations (Eq. (18)) indicate that the energy-momentum tensor is vanished and this result is inconsistent with some non-trivial matter configurations. This problem is the main obstacle which the physicists face with. Then, what is the constraint which will define the metric of the 2 dimensional space-time in the presence

of the gravitational field? As an answer, many interesting suggestions are advanced [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. One of these suggestions has shown that classical gravity in two spacetime dimensions need not be so trivial [15], and that an interesting relativistic theory of gravitation in this context may be formed by setting

$$R = 8\pi GT = 8\pi GT^\mu_\mu, \quad (24)$$

$T = T^\mu_\mu$ representing the trace of the conserved energy-momentum tensor. It is shown [29] that this theory is a 2D limit of the 4D General Relativity. In spite of its simplicity, this approach has many interesting classical and semi-classical results, including a well-defined Newtonian limit [15], black holes [16, 17], a post-Newtonian expansion, gravitational waves, FRW cosmologies, gravitational collapse [18] and black hole radiation [18, 19, 20]. The aim of the present paper is not to define the metric of the 2D spaces, but is to construct a theory of a quantum gravity in the 2D spaces. Such a theory will permit to have an idea about the manner with which we should construct the 4D theory of quantum gravity.

3. Gravitational Quantum Stationary Hamilton-Jacobi Equation

a. general form of the GQHJE

As we have mentioned above, in the present paper, we restrict our study on the 2D spaces ie spaces with the temporal coordinate t and one spatial coordinate x . In this section we derive the (1+1)D Gravitational Quantum Hamilton-Jacobi Equation (GQHJE) from the (1+1)D Klein-Gordon equation written in the case of the presence of a gravitational field. The considered Klein-Gordon equation is given in Ref. [32] as

$$\frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} g^{\alpha\beta} \partial_\beta) \Psi(X) + \frac{m_o^2 c^2}{\hbar^2} \Psi(X) = 0, \quad \alpha, \beta = 0, 1. \quad (25)$$

$\Psi(X)$ is the wave function, and $X := (t, x)$. $g^{\alpha\beta}(X)$ are the components of the inverse of the metric tensor $g(X)$. $|g|$ represents the determinant's

positive value of the metric tensor. Eq. (25) can be written as

$$g^{\alpha\beta} \partial_{\alpha\beta}^2 \Psi + \partial_{\alpha} (\sqrt{|g|} g^{\alpha\beta}) \partial_{\beta} \Psi + \frac{m_o^2 c^2}{\hbar^2} \Psi = 0, \quad \alpha, \beta = 0, 1, \quad (26)$$

where $\partial_{\alpha\beta}^2$ is the second order derivative with respect to x^{α} and x^{β} .

Now, we derive the GQHJE. Let us write the wave function Ψ with the Bohm-de Broglie notation

$$\Psi(X) = A(X) \cdot \exp\left(\frac{i}{\hbar} S(X)\right), \quad (27)$$

where $A(X)$ and $S(X)$ are real functions representing the amplitude and the phase of the wave function. Replacing Eq. (27) into Eq. (26) and separating the imaginary and real parts of the resulting equation, we get

$$g^{\alpha\beta} \partial_{\alpha} S \partial_{\beta} S - \frac{\hbar^2}{A} g^{\alpha\beta} \partial_{\alpha\beta}^2 A - \frac{\hbar^2}{A \sqrt{|g|}} \cdot \partial_{\alpha} (\sqrt{|g|} g^{\alpha\beta}) \partial_{\beta} A - m_o^2 c^2 = 0 \quad (28)$$

and

$$g^{\alpha\beta} (\partial_{\beta} A \partial_{\alpha} S + \partial_{\alpha} A \partial_{\beta} S + A \partial_{\alpha\beta}^2 S) + \frac{1}{\sqrt{|g|}} \partial_{\alpha} (\sqrt{|g|} g^{\alpha\beta}) A \partial_{\beta} S = 0 \quad (29)$$

The two last equations represent the GQHJE for a general metric and non-stationary case. Note that when we take the limit $\hbar \rightarrow 0$ Eq. (28) reduces to the Hamilton-Jacobi equation written in the presence of a gravitational field [30, 31]

$$g^{\alpha\beta} \partial_{\alpha} S \partial_{\beta} S = m_o^2 c^2. \quad (30)$$

b. The GQSHJE for a static metric and stationary case

Now, we investigate the static metric and the stationary case. The general form of a static metric is given by ($g_{10} = g_{01} = 0$)

$$g = \begin{pmatrix} g_{00}(x) & 0 \\ 0 & g_{11}(x) \end{pmatrix}; \quad g^{-1} = \begin{pmatrix} g^{00}(x) & 0 \\ 0 & g^{11}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{g_{00}(x)} & 0 \\ 0 & \frac{1}{g_{11}(x)} \end{pmatrix}, \quad (31)$$

while for the stationary system we have

$$S(X) = S_0(x) - E t; \quad A(X) = A(x). \quad (32)$$

Replacing Eqs. (31) and (32) into Eqs. (28) and (29), we find after devising by $2m_o$

$$\begin{aligned} \frac{1}{2m_o} \left(\frac{\partial S_0}{\partial x} \right)^2 \frac{1}{g_{11}} - \frac{\hbar^2}{2m_o g_{11}} \frac{1}{A} \frac{\partial^2 A}{\partial x^2} - \frac{\hbar^2}{2m_o \sqrt{|g|}} \cdot \\ \cdot \frac{\partial}{\partial x} \left(\frac{\sqrt{|g|}}{g_{11}} \right) \frac{1}{A} \frac{\partial A}{\partial x} + \frac{1}{2m_o c^2} \left(\frac{E^2}{g_{oo}} - m_o^2 c^4 \right) = 0 \end{aligned} \quad (33)$$

and

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x} \left[\sqrt{|g|} \frac{1}{g_{11}} A^2 \frac{\partial S_0}{\partial x} \right] = 0. \quad (34)$$

Resolving EQ. (34), we find

$$A(x) = k \left(\frac{\sqrt{|g|}}{g_{11}} \frac{\partial S_0}{\partial x} \right)^{-\frac{1}{2}} \quad (35)$$

with which after replacing into Eq. (33) we get

$$\begin{aligned} \frac{1}{2m_o} \left(\frac{\partial S_0}{\partial x} \right)^2 \frac{1}{g_{11}} - \frac{\hbar^2}{4m_o g_{11}} \{S_0, x\} + \frac{\hbar^2}{2m_o \sqrt{g_{11}} \sqrt{|g|}} \cdot \\ \cdot \frac{\partial^2}{\partial x^2} \left(\sqrt{\frac{|g|}{g_{11}}} \right) + \frac{1}{2m_o c^2} \left(\frac{E^2}{g_{oo}} - m_o^2 c^4 \right) = 0. \end{aligned} \quad (36)$$

This last equation represents the GQSHJE for the static metric and the stationary systems. We remark that after taking the limit $\hbar \rightarrow 0$ into Eq. (36), we find

$$\frac{1}{2m_o} \left(\frac{\partial S_0}{\partial x} \right)^2 \frac{1}{g_{11}} + \frac{1}{2m_o c^2} \left(\frac{E^2}{g_{oo}} - m_o^2 c^4 \right) = 0. \quad (37)$$

It is clear that Eq. (37) is the classical Hamilton-Jacobi equation in the presence of a gravitational field and represents the particular case of Eq. (30) (Stationary system and static metric).

Note also that if we consider a vanishing gravitational field ($g_{oo} = 1$ and $g_{11} = -1$), Eq. (36) goes to the RQSHJE (Eq. (7)).

c. The solution of the GQSHJE

As a solution of the GQSHJE (Eq. (36)) we propose the expression (4) of the reduced action, but where $\theta(x)$ and $\phi(x)$ represent the two real and independent solutions of the Klein-Gordon equation describing the case of a static metric and a stationary systems

$$\frac{\hbar^2 c^2}{g_{11}} \frac{\partial^2 \psi(x)}{\partial x^2} + \frac{\hbar^2 c^2}{\sqrt{|g|}} \frac{\partial \psi(x)}{\partial x} \frac{\partial}{\partial x} \left(\frac{\sqrt{|g|}}{g_{11}} \right) + \left(m_o^2 c^4 - \frac{E^2}{g_{oo}} \right) \psi(x) = 0, \quad (38)$$

from which we deduce that the wronwkian of θ and ϕ verifies

$$W = \phi \frac{d\theta}{dx} - \theta \frac{d\phi}{dx} = \lambda \frac{g_{11}}{\sqrt{|g|}}. \quad (39)$$

Now, we verify that expression (4) is the solution of Eq. (36). In this order, let us write Eq. (4) in the following form

$$S_0(x) = \hbar \arctan \left(\frac{\theta'}{\phi} \right), \quad (40)$$

where

$$\theta' = a\theta + b\phi.$$

Because it is a linear combination of θ and ϕ , θ' is a solution of Eq. (38). Taking the first, the second and the third derivatives of S_0 with respect to x from Eq. (40), then replacing theme into Eq. (36), we find

$$\frac{\hbar^2}{g_{11}} \frac{\left[\left(\lambda \frac{g_{11}}{\sqrt{|g|}} \right)^2 - W^2 \right]}{(\theta'^2 + \phi^2)^2} - \frac{\theta'}{(\theta'^2 + \phi^2)} \left[\frac{\hbar^2}{g_{11}} \frac{\partial^2 \theta'}{\partial x^2} + \frac{\hbar^2}{\sqrt{|g|}} \right].$$

$$\begin{aligned}
& \cdot \frac{\partial}{\partial x} \left(\frac{\sqrt{|g|}}{g_{11}} \right) \frac{\partial \theta'}{\partial x} + \frac{1}{c^2} \left(m_o^2 c^4 - \frac{E^2}{g_{oo}} \right) \Big] - \frac{\phi}{(\theta'^2 + \phi^2)} \cdot \\
& \cdot \left[\frac{\hbar^2}{g_{11}} \frac{\partial^2 \phi}{\partial x^2} + \frac{\hbar^2}{\sqrt{|g|}} \frac{\partial}{\partial x} \left(\frac{\sqrt{|g|}}{g_{11}} \right) \frac{\partial \phi}{\partial x} + \frac{1}{c^2} \left(m_o^2 c^4 - \frac{E^2}{g_{oo}} \right) \right] = 0. \quad (41)
\end{aligned}$$

Taking account of Eq. (39) and since θ' and ϕ are solutions of Eq. (38), Eq. (41) is automatically satisfied. This demonstrate that the expression (4) of the reduced action is the solution of the GQSHJE (Eq. (36)).

4. Gravitational Quantum Law of Motion for a stationary case and a static metric

Before studying the dynamical behaviour of a quantum particle moving under the gravitational field, we introduce a quantum gravitational transformation of the coordinate $x \rightarrow \hat{x}$ with which the GQSHJE giving in Eq. (36) takes the form of the classical gravitational Hamilton-Jacobi equation (Eq. (37)) with respect to the new coordinate \hat{x} . Such a transformation is defined as

$$\begin{aligned}
\left(\frac{\partial x}{\partial \hat{x}} \right)^2 &= \left\{ 1 - \frac{\hbar^2}{2} \left(\frac{\partial S_0}{\partial x} \right)^{-2} \left[\{S_o, x\} - 2 \sqrt{\frac{g_{11}}{\sqrt{|g|}}} \frac{\partial^2}{\partial x^2} \left(\sqrt{\frac{\sqrt{|g|}}{g_{11}}} \right) \right] \right\} \\
&= \frac{g_{11}}{c^2} \left(\frac{\partial S_0}{\partial x} \right)^{-2} \left(m_o^2 c^4 - \frac{E^2}{g_{oo}} \right). \quad (42)
\end{aligned}$$

Here, we want to underline that the notion of a gravitational quantum coordinate \hat{x} is extrapolated from the notion of the quantum coordinate already introduced by Faraggi and Matone in Refs. [9, 10].

Now, in order to derive the Gravitational quantum Law of motion, let us introduce a Lagrangian of the form

$$L(x, \dot{x}, a, b) = -m_0 c^2 \sqrt{g_{oo}(x) + g_{11}(x) f(x, a, b) \frac{\dot{x}^2}{c^2}}, \quad (43)$$

Where a and b are two constants of integration given in Eq. (4). f is a real

function given by

$$f(x, a, b) = \left(\frac{\partial x}{\partial \hat{x}} \right)^{-2} = \frac{c^2}{g_{11}} \left(\frac{\partial S_0}{\partial x} \right)^2 \left(m_o^2 c^4 - \frac{E^2}{g_{oo}} \right)^{-1}. \quad (44)$$

Using this expression of the Lagrangian, the least action principle leads to

$$-\frac{m_o c^2}{2} \left(1 + \frac{g_{11} f \dot{x}^2}{g_{oo} c^2} \right)^{-\frac{3}{2}} \sqrt{g_{oo}} \frac{d}{dt} \left(\frac{\dot{x}^2 g_{11} f}{c^2 g_{oo}} \right) + m_o c^2 \left(1 + \frac{g_{11} f \dot{x}^2}{g_{oo} c^2} \right)^{-\frac{1}{2}} \frac{d}{dt} (\sqrt{g_{oo}}) = 0, \quad (45)$$

which, after integration, gives

$$E = \frac{m_o c^2 g_{oo}(x)}{\sqrt{g_{oo}(x) + g_{11}(x) f(x, a, b) \frac{\dot{x}^2}{c^2}}}. \quad (46)$$

E is an integration constant representing the total energy of the quantum particle moving in the presence of a gravitational field. Note that in the case of a vanishing gravitational field ($g_{oo} = 1$ and $g_{11} = -1$) and at the classical limit ($\hbar \rightarrow 0, f(x, a, b) \rightarrow 1$), Eq. (46) reduces to the conservation equation

$$E = \frac{m_o c^2}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}} \quad (47)$$

well known in special relativity. Remark that taking the expression of $f(x, a, b)$ from Eq. (44) in Eq. (46), we deduce the relation

$$\dot{x} \cdot \frac{\partial S_0}{\partial x} = E - \frac{m_o^2 c^4}{E} g_{oo}(x) \quad (48)$$

connecting between the momentum $\partial S_0 / \partial x$ and the velocity of the particle. This relation represents the gravitational quantum law of motion. Indeed, once the metric (g_{oo} and g_{11}) of the 2D spacetime is determined, we can plot the trajectories of the particle from Eq. (48) after determining the solutions θ and ϕ of the Klein-Gordon equation (Eq. (38)). For a vanishing gravitational field Eq. (48) reduces to Eq. (10) ($V(x)$ being vanish) representing the relativistic quantum law of motion.

5. Hilbert-Einstein action and the GQ Law of motion

In the present section, we derive the GQ law of motion of a quantum particle moving under the gravitational field in the general case (ie the field is not static and the system is not stationary). For this aim, let us introduce the Hilbert-Einstein action

$$S(x^\mu) = -m_0 c \int ds = -m_0 c \int \sqrt{G_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} ds , \quad (49)$$

where

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu , \quad (50)$$

and $G_{\mu\nu}(x^\alpha)$ are the components of the gravitational quantum metric's tensor, and $\dot{x}^\mu = dx^\mu/ds$. Writing expression (49) of the action as follows

$$S(x^\mu) = \int L(x^\mu, \dot{x}^\mu) ds , \quad (51)$$

one can define the Lagrangian as

$$L(x^\mu, \dot{x}^\mu) = -m_0 c \sqrt{G_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} . \quad (52)$$

Using expression (52) of the Lagrangian, the least action principle leads to

$$\frac{d^2 x^\sigma}{ds^2} + \Gamma_{\alpha\beta}^\sigma \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 , \quad \alpha, \beta = 0, 1 . \quad (53)$$

where the $\Gamma_{\mu\nu}^\gamma$ are the Christoffel symbols given in Eq. (20), but where the metric tensor $g_{\mu\nu}$ is replaced by $G_{\mu\nu}$. The action (49) is invariant under reparametrization $s \rightarrow f(s)$. This gauge symmetry leads to the constrained dynamics in the Hamiltonian formulation [31]. The constraint reads

$$G^{\mu\nu} p_\mu p_\nu - m_0^2 c^2 = 0 , \quad (54)$$

where $p_\mu = \partial L / \partial \dot{x}^\mu$ are canonical momenta.

In the Hamilton-Jacobi theory the canonical momenta are defined as

$$p_\mu = \frac{\partial S}{\partial x^\mu} . \quad (55)$$

Replacing Eq. (55) into Eq. (54), we get

$$G^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} - m_0^2 c^2 = 0 . \quad (56)$$

Eq. (56) represents the Hamilton-Jacobi equation for a quantum particle moving under the gravitational field. It must be equivalent to Eq. (28). After comparison of Eqs. (28) and (56), we deduce that the components of the metric tensor $G^{\mu\nu}$ must be written as

$$G^{\mu\nu} = g^{\mu\nu} f^{(\mu\nu)} \quad (57)$$

Where $g^{\mu\nu}$ are the components of the metric tensor in the missing of the quantum potential ($\hbar \rightarrow 0$), and

$$f^{(\mu\nu)} = \left[1 - \frac{\hbar^2}{A} (\partial_\mu S \partial_\nu S)^{-1} \left(\partial_{\mu\nu}^2 A + \frac{1}{g^{\mu\nu} \sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu}) \partial_\nu A \right) \right] \quad (58)$$

are the components of the metric tensor in the presence of a quantum potential and the missing of the gravitational interaction. It is useful to underline that the right hand side of Eq. (57) do not support a summation over μ and ν , it is just a product of the components $g^{\mu\nu}$ and $f^{(\mu\nu)}$. We note also that there is no summation in the right hand side of Eq. (58). It is clear that when we take the classical limit ($\hbar \rightarrow 0$) the components $G^{\mu\nu}$ reduce to $g^{\mu\nu}$.

Now, if one consider the stationary system and a static metric of the space, he will find the results obtained in Sec. 4. So, the expression (52) of the Lagrangian reduces to the expression (43), and the dynamical equation (53) reduces to Eq. (45). In this case the components $G^{\mu\nu}$ (Eq. (57)) reduce to

$$\begin{aligned} G^{\mu\nu} &= \begin{pmatrix} \frac{1}{g_{oo}} & 0 \\ 0 & \left\{ \frac{1}{g_{11}} - \frac{1}{g_{11}} \frac{\hbar^2}{2} \left(\frac{\partial S_0}{\partial x} \right)^{-2} \left[\{S_o, x\} - 2 \sqrt{\frac{g_{11}}{\sqrt{|g|}}} \frac{\partial^2}{\partial x^2} \left(\sqrt{\frac{\sqrt{|g|}}{g_{11}}} \right) \right] \right\} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{g_{oo}} & 0 \\ 0 & \frac{1}{g_{11} f(x,a,b)} \end{pmatrix} , \end{aligned} \quad (59)$$

where $f(x, a, b)$ is given in Eq. (44). Replacing expression (59) of the metric tensor into Eq. (52), the Lagrangian takes the form given in Eq. (43). Then, using Eq. (55) of the canonical momenta and taking into account $p_\mu = \partial L / \partial \dot{x}^\mu$, we get

$$\frac{\partial S}{\partial x^0} = \frac{1}{c} \frac{\partial S}{\partial t} = -\frac{E}{c} = \frac{\partial L}{\partial \dot{x}^0} = -\frac{m_o c^2 g_{oo} \frac{dt}{ds}}{\sqrt{g_{00} c^2 \frac{dt}{ds}^2 + g_{11} f(x, a, b) \dot{x}^2}}, \quad (60)$$

which can be written as

$$E = \frac{m_o c^2 g_{oo}}{\sqrt{g_{00} + g_{11} f(x, a, b) \left(\frac{\dot{x}}{c}\right)^2}}.$$

The last equation is equivalent to Eq. (46). It represents the expression of the total energy of the particle for a stationary system and a static metric.

The fact that the dynamical equations describing the motion of a quantum particle under gravitation in the general case reduce to the those describing the motion in the case of a stationary system and a static metric is an important results. It indicates that the generalization for a general system that we presented in this section is correct, so the GQ law of motion given in this section can be considered as the **Universal Law of Motion**. Thus, in 2D, the geodesic equation (53) describes the trajectories of a quantum particle moving under gravitation in a space with a general metric and for a non-stationary systems. Such a system has a Lagrangian of the form (52) and verify the GQHJE (Eq. (28)).

6. Conclusion

First, we would like to underline that in this article we do not bring a description of the geometry of the 2D curved space under gravitation. In other words, we do not specify the form of the metric tensor $g_{\mu\nu}$ of the 2D timespace, we just exposed the manner with which the 2D general relativity should be connected with the 2D deterministic approach of quantum mechanics presented in Refs. [1, 2, 3, 4, 5, 6, 7, 8].

In the present paper, there is four main results. The first one is the establishment of the GQHJE for general metric and non-stationary case (Eqs. (28) and (29)). The second result is the establishment of the GQSHJE (Eq. (36)) for the stationary case and a static metric. The third result is the elaboration of the gravitational quantum law of motion for static metric and stationary system (Eqs. (43), (46) and (48)). The forth result is the generalization of the GQ law of motion into general metric and non-stationary system (Eqs. (52), (53), (54) and (56)). Note that when we take the limit $\hbar \rightarrow 0$, all the established equations reduce to those of the general relativity (GR), and when we consider a vanishing gravitational potential, all the established equations reduce to those of the Relativistic quantum mechanics [4, 5].

We would like to stress that the introduction of the gravitation in our deterministic approach of quantum mechanics in 2-dimensions is an important and a daring step to connect the GR and the deterministic quantum mechanics especially for the 4D spacetime. We think that in order to generalize our deterministic approach of quantum gravity into 4D, it is primordial to use the Einstein-Hilbert action (Eqs. (49) and (51)). This suggestion will be exposed in a preparing paper.

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